ON EQUATIONS FOR THE POSITION DETERMINATION OF A MOVING OBJECT

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The present paper is an outgrowth of [1 and 2] in the parts concerning the integration of the equations determining the coordinates of an object moving over the Earth's globe.

1. The problem of the autonomous determination of the coordinates of an object moving over the surface of the Earth may be solved by different methods.

Two versions of inertial systems are considered in [1 and 2]. In both these versions, the problem of integrating a system of nonlinear differential equations of the form

$$(U+\lambda^{\cdot})\cos\varphi\sin\vartheta-\varphi^{\cdot}\cos\vartheta=\omega_x(t),\qquad (U+\lambda^{\cdot})\cos\varphi\cos\vartheta+\varphi^{\cdot}\sin\vartheta=\omega_y(t)$$

$$(U+\lambda^{\cdot})\sin\varphi+\vartheta^{\cdot}=\omega_z(t) \qquad (1.1)$$

is imposed on some computer unit.

Here ω_x, ω_y, ω_z should be considered known functions of t.

Let us keep in mind the second of the above-mentioned versions of the inertial system which consists of a space gyrocompass, a directional gyroscope and an integrator [2]. In this case, as has been shown in [2 and 3], we can always consider $\omega_x\equiv 0$, and we then have from (1.1)

$$\varphi^{\cdot}=\omega_y\sin\vartheta,\qquad \vartheta^{\cdot}=\omega_z-\omega_y\cos\vartheta\tan\varphi,\qquad \lambda^{\cdot}=-U+\omega_y\cos\vartheta\sec\varphi$$

$$\left(\vartheta=\tan^{-1}\frac{v_N}{RU\cos\varphi+v_E}\right) \qquad (1.2)$$

The notation in (1.2) is the same as in [2 and 3]. Let us recall that φ and λ are the present latitude and longitude of the position; ϑ is the present value of the velocity deviation of the gyro-horizon compass. Here v_N and v_E are the northerly and easterly components, respectively, of the object's velocity relative to the Earth. In the system under consideration the velocity correction to the compass is produced autonomously.

2. The method of successive approximations in some form or other is customarily used in investigations of systems of the type (1.1). For example, representing the first two equations of (1.2) as a system of integral equations we obtain

$$\varphi = \varphi(0) + \int_0^t \omega_y(\tau) \sin \vartheta(\tau)\, d\tau, \qquad \vartheta = \vartheta(0) + \int_0^t \omega_z(\tau)\, d\tau - \int_0^t \omega_y(\tau) \cos\vartheta(\tau) \tan\varphi(\tau)\, d\tau \tag{2.1}$$

Here $\varphi(0)$ and $\vartheta(0)$ are the initial values of the latitude of the position and of the velocity deviation.

According to the method of successive approximations we can assume

$$\varphi^{(n)} = \varphi^{(0)} + \int_0^t \omega_y(\tau) \sin\vartheta^{(n-1)}(\tau)\, d\tau, \quad \vartheta^{(n)} = \vartheta^{(0)} - \int_0^t \omega_y(\tau) \cos\vartheta^{(n-1)}(\tau) \tan\varphi^{(n-1)}(\tau)\, d\tau$$

$$\left(\varphi^{(0)} \equiv \varphi(0), \quad \vartheta^{(0)} \equiv \vartheta(0) + \int_0^t \omega_z(\tau)\, d\tau\right) \tag{2.2}$$

Use of this method on the system (2.1) is known to be inconvenient, in that the actual realization of the quadratures is rapidly made complicated as the approximations are evaluated. Moreover, strictly speaking the range of variation of t within which the solution of the system (2.1) may be obtained by the Pickard method should be established.

3. Let us assume in (1.1)

$$\lambda + Ut = \psi, \qquad {}^1/_2\pi - \varphi = \theta \tag{3.1}$$

and let ϑ be denoted by φ_0. We then arrive at equations of the form

$$\psi^\cdot \sin\theta \sin\varphi_0 + \theta^\cdot \cos\varphi_0 = \omega_x, \quad \psi^\cdot \sin\theta\cos\varphi_0 - \theta^\cdot \sin\varphi_0 = \omega_y, \quad \psi^\cdot \cos\theta + \varphi_0^\cdot = \omega_z \tag{3.2}$$

The system (3.2) has the structure of the Euler kinematic equations in the variables ψ, θ and φ_0. Hence, the problem of integrating the system (1.1) for given ω_x, ω_y and ω_z is equivalent to the classical problem of determining the position of a solid body by means of its angular velocity. In the general case this problem, which reduces to the Riccati equation, does not reduce to quadratures [4]. Putting $\omega_x = 0$ in (3.2) and, moreover,

$$\gamma_1 = \sin\varphi_0 \sin\theta, \qquad \gamma_2 = \cos\varphi_0 \sin\theta, \qquad \gamma_3 = \cos\theta \tag{3.3}$$

we obtain the Poisson equation [5] in the variables γ_1, γ_2, γ_3

$$\gamma_1^\cdot = \omega_z\gamma_2 - \omega_y\gamma_3, \qquad \gamma_2^\cdot = -\omega_z\gamma_1, \qquad \gamma_3^\cdot = \omega_y\gamma_1 \tag{3.4}$$

The Rodrigue-Hamilton equations for the case under consideration will be

$$2\lambda_0^\cdot = -\omega_y\lambda_2 - \omega_z\lambda_3, \qquad 2\lambda_1^\cdot = \omega_z\lambda_2 - \omega_y\lambda_3$$
$$2\lambda_2^\cdot = \omega_y\lambda_0 - \omega_z\lambda_1, \qquad 2\lambda_3^\cdot = \omega_z\lambda_0 + \omega_y\lambda_1 \tag{3.5}$$

where

$$\lambda_0 = \cos\frac{\theta}{2}\cos\frac{\psi+\varphi_0}{2}, \quad \lambda_1 = \sin\frac{\theta}{2}\cos\frac{\psi-\varphi_0}{2}$$
$$\lambda_2 = \sin\frac{\theta}{2}\sin\frac{\psi-\varphi_0}{2}, \quad \lambda_3 = \cos\frac{\theta}{2}\sin\frac{\psi+\varphi_0}{2} \tag{3.6}$$

in which the parameters λ_s satisfy the condition [4]

$$\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1 \tag{3.7}$$

Let us represent (3.4) and (3.5) as equivalent systems of integral equations

$$\gamma_1 = \gamma_1(0) + \int_0^t (\omega_z\gamma_2 - \omega_y\gamma_3)\, d\tau, \quad \gamma_2 = \gamma_2(0) - \int_0^t \omega_z\gamma_1\, d\tau, \quad \gamma_3 = \gamma_3(0) + \int_0^t \omega_y\gamma_1\, d\tau \tag{3.8}$$

$$\lambda_0 = \lambda_0(0) - \frac{1}{2}\int_0^t (\omega_y\lambda_2 + \omega_z\lambda_3)\,d\tau, \qquad \lambda_1 = \lambda_1(0) + \frac{1}{2}\int_0^t (\omega_z\lambda_2 - \omega_y\lambda_3)\,d\tau$$
$$\lambda_2 = \lambda_2(0) + \frac{1}{2}\int_0^t (\omega_y\lambda_0 - \omega_z\lambda_1)\,d\tau, \qquad \lambda_3 = \lambda_3(0) + \frac{1}{2}\int_0^t (\omega_z\lambda_0 + \omega_y\lambda_1)\,d\tau \tag{3.9}$$

Here, the $\gamma_s(0)$ are initial values of the variables γ_s, $\lambda_s(0)$ are the initaial values of the Rodrigue-Hamilton parameters. Because of the linearity of the systems (3.8) and (3.9), their symmetry and the absence of trigonometric functions of the unknowns, successive approximations are more effective here for the system (1.1). In a first approximation we have from (3.8) and (3.9)

$$\gamma_1^{(1)} = \gamma_1(0) + \gamma_2(0)\,\Omega_z(t) - \gamma_3(0)\,\Omega_y(t)$$
$$\gamma_2^{(1)} = \gamma_2(0) - \gamma_1(0)\,\Omega_z(t) \qquad \left(\Omega_{y,z}(t) = \int_0^t \omega_{y,z}(\tau)\,d\tau\right) \tag{3.10}$$
$$\gamma_3^{(1)} = \gamma_3(0) + \gamma_1(0)\,\Omega_y(t)$$

and also

$$\lambda_0^{(1)} = \lambda_0(0) - {}^1/_2\,\lambda_2(0)\,\Omega_y(t) - {}^1/_2\,\lambda_3(0)\,\Omega_z(t)$$
$$\lambda_1^{(1)} = \lambda_1(0) + {}^1/_2\,\lambda_2(0)\,\Omega_z(t) - {}^1/_2\,\lambda_3(0)\,\Omega_y(t) \tag{3.11}$$
$$\lambda_2^{(1)} = \lambda_2(0) + {}^1/_2\,\lambda_0(0)\,\Omega_y(t) - {}^1/_2\,\lambda_1(0)\,\Omega_z(t)$$
$$\lambda_3^{(1)} = \lambda_3(0) + {}^1/_2\,\lambda_0(0)\,\Omega_z(t) + {}^1/_2\,\lambda_1(0)\,\Omega_y(t)$$

The values of λ_s determined by Formulas (3.11), satisfy condition (3.7) if we limit ourselves to first order terms in $\Omega_y(t)$ and $\Omega_z(t)$.

4. Using (3.10) and taking (3.3) into account, we obtain

$$\cos\theta = \cos\theta(0) + \sin\varphi_0(0)\sin\theta(0)\,\Omega_y(t)$$
$$\tan\varphi_0 = \frac{\sin\varphi_0(0)\sin\theta(0) + \cos\varphi_0(0)\sin\theta(0)\,\Omega_z(t) - \cos\theta(0)\,\Omega_y(t)}{\cos\varphi_0(0)\sin\theta(0) - \sin\varphi_0(0)\sin\theta(0)\,\Omega_z(t)} \tag{4.1}$$

Returning to the old variables by means of (4.1), we have

$$\sin\varphi = \sin\varphi(0) + \cos\varphi(0)\sin\vartheta(0)\,\Omega_y(t)$$
$$\tan\vartheta = \frac{\sin\vartheta(0) + \cos\vartheta(0)\,\Omega_z(t) - \tan\varphi(0)\,\Omega_y(t)}{\cos\vartheta(0) - \sin\vartheta(0)\,\Omega_z(t)} \tag{4.2}$$

For the longitude we have from the third equation of the system (1.2)

$$\lambda = \lambda(0) - Ut + \int_0^t \omega_y(\tau)\cos\vartheta(\tau)\sec\varphi(\tau)\,d\tau \tag{4.3}$$

Here φ and ϑ are determined by means of Expressions (4.2). By substituting their initial values for ϑ and φ in (4.3) we have for the first estimate

$$\lambda = \lambda(0) - Ut + \cos\vartheta(0)\sec\varphi(0)\,\Omega_y(t) \tag{4.4}$$

A more exact formula may be obtained for λ by using the Rodrigue-Hamilton parameters. From (3.6) we have

$$\tan\psi = \frac{\lambda_1\lambda_3 + \lambda_0\lambda_2}{\lambda_0\lambda_1 - \lambda_2\lambda_3} \tag{4.5}$$

Using (3.11) and being limited only to first order parameters in the integrals of ω_y and ω_z we have

$$\lambda_0\lambda_1 - \lambda_2\lambda_3 = \lambda_0(0)\lambda_1(0) - \lambda_2(0)\lambda_3(0) - [\lambda_0(0)\lambda_3(0) + \lambda_1(0)\lambda_2(0)]\,\Omega_y(t)$$
$$\lambda_1\lambda_3 + \lambda_0\lambda_2 = \lambda_1(0)\lambda_3(0) + \lambda_0(0)\lambda_2(0) + {}^1/_2\,[\lambda_0^2(0) + \lambda_1^2(0) - \lambda_2^2(0) - \lambda_3^2(0)]\,\Omega_y(t) \qquad (4.6)$$

By virtue of (3.6), here

$$\lambda_0(0)\lambda_1(0) - \lambda_2(0)\lambda_3(0) = {}^1/_2 \sin\theta(0)\cos\psi(0)$$
$$\lambda_1(0)\lambda_3(0) + \lambda_0(0)\lambda_2(0) = {}^1/_2 \sin\theta(0)\sin\psi(0)$$
$$\lambda_0(0)\lambda_3(0) + \lambda_1(0)\lambda_2(0) = {}^1/_2\,[\sin\psi(0)\cos\varphi_0(0) + \cos\psi(0)\sin\varphi_0(0)\cos\theta(0)] \qquad (4.7)$$
$$\lambda_0^2(0) + \lambda_1^2(0) - \lambda_2^2(0) - \lambda_3^2(0) = \cos\psi(0)\cos\varphi_0(0) - \sin\psi(0)\sin\varphi_0(0)\cos\theta(0)$$

Hence

$$\tan\psi = \frac{\sin\theta(0)\sin\psi(0) + [\cos\psi(0)\cos\varphi_0(0) - \sin\psi(0)\sin\varphi_0(0)\cos\theta(0)]\;\Omega_y(t)}{\sin\theta(0)\cos\psi(0) - [\sin\psi(0)\cos\varphi_0(0) + \cos\psi(0)\sin\varphi_0(0)\cos\theta(0)]\;\Omega_y(t)} \qquad (4.8)$$

Returning to the old variables, we finally have (4.9)

$$\lambda = -Ut + \tan^{-1}\frac{\sin\lambda(0) + [\cos\lambda(0)\cos\vartheta(0)\sec\varphi(0) - \sin\lambda(0)\sin\vartheta(0)\tan\varphi(0)]\,\Omega_y(t)}{\cos\lambda(0) - [\sin\lambda(0)\cos\vartheta(0)\sec\varphi(0) + \cos\lambda(0)\sin\vartheta(0)\tan\varphi(0)]\,\Omega_y(t)}$$

5. The accuracy with which this inertial system produces the coordinates of the position in the presence of errors in the gyrocompass and directional gyroscope readings may be estimated by means of (4.2), (4.3) and (4.9). In particular, the expressions

$$\omega_y = v/R + \omega\beta + \gamma^{\cdot}, \qquad \omega_z = -v/R\beta + \omega + \alpha^{\cdot}$$

where α, β and γ are angles defining the position of the gyrocompass [3], should be taken in the mentioned formulas for ω_y and ω_z when it is necessary to take account of small gyrocompass motions (which prctically always exist).

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